FORMALITY CONJECTURE

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1. Introduction

This paper is devoted to a conjecture concerning the deformation quantization. This conjecture implies that arbitrary smooth Poisson manifold can be formally quantized, and the equivalence class of the resulting algebra is canonically defined. In other terms, it means that non-commutative geometry, in the formal approximation to the commutative geometry of smooth spaces, is described by the semi-classical approximation.

Recently an article by A. Voronov (see [V]) with the exposition of the formality conjecture appeared on the net. The present paper can be seen as a companion to [V]. Here I present the conjecture in a slightly different form. In order to do it I include some preparational material on deformation theory and homotopy theory of differential graded Lie algebras. Brute force calculations confirm (locally) the conjecture up to the 6-th order in the perturbation theory. As a by-product I obtained a formula for a new flow on the space of germs of Poisson manifolds. Also I propose a reformulation of the conjecture and further evidences.

1.1. Deformation quantization

Let A be the algebra of smooth functions on a C^∞-manifold X. We are interested in star-products on A (see [BBFLS]), i.e. associative $\mathbf{R}[[\hbar]]$-linear products on $A[[\hbar]]$ given by formulas

$$(f,g)\mapsto f*g = fg + \hbar B_1(f,g) + \hbar^2 B_2(f,g) + \ldots \in A[[\hbar]]$$

where $\hbar$ is the formal variable, and B_i are bi-differential operators. There is a natural gauge group acting on star-products. It acts via linear transformations $A\longrightarrow A$ parametrized by $\hbar$:

$$f\mapsto f + \hbar D_1(f) + \hbar^2 D_2(f) + \ldots$$

where D_i are differential operators. The formality conjecture will give a clear picture of star-products. In particular, for any Poisson bracket B_1, there should be a canonical gauge equivalence class of star-products with B_1 as the first term.

2. Deformation theory via differential graded Lie algebras

This part is essentially standard (see [GM], [HS1], [SS], ...).

Let $\mathbf{g}$ be a differential graded Lie algebra over a field of characteristic zero:

$$\mathbf{g} = \bigoplus_{k\in\mathbf{Z}} \mathbf{g}^k,\quad [\ ,\ ]:\mathbf{g}^k\otimes\mathbf{g}^l\longrightarrow\mathbf{g}^{k+l},\quad d:\mathbf{g}^k\longrightarrow\mathbf{g}^{k+1},\quad d^2=0\quad .$$

In other words, $\mathbf{g}$ is a Lie algebra in the tensor category of complexes of vector spaces.

We associate with it a functor Def on finite-dimensional commutative associative algebras over the same ground field with values in the category of sets.

First of all, let us assume that $\mathbf{g}$ is a nilpotent Lie superalgebra. We define set $\mathcal{M}(\mathbf{g})$ (the set of solutions of the Maurer-Cartan equation) by the formula

$$\mathcal{M}(\mathbf{g}) := \left\{\gamma\in\mathbf{g}^1 \mid\, d\gamma + \frac{1}{2}[\gamma,\gamma]=0\right\}/\Gamma^0$$

where Γ^0 is the nilpotent group associated with the nilpotent Lie algebra $\mathbf{g}^0$. The action of Γ^0 can be defined by the exponentiation of the infinitesimal action of its Lie algebra:

$$\alpha \in \mathbf{g}^0 \mapsto (\dot{\gamma} = d\alpha + [\alpha, \gamma]) \ .$$

Now we are ready to define the functor Def. Technically, it is convenient to define it on the category of finite-dimensional nilpotent commutative associative algebras without unit. Let $\mathbf{m}$ be such an algebra, $\mathbf{m}^{dim\, \mathbf{m}+1} = 0$. The functor is given (on objects) by the formula

$$Def(\mathbf{m}) = \mathcal{M}(\mathbf{g} \otimes \mathbf{m}) \ .$$

In the conventional approach $\mathbf{m}$ is the maximal ideal in a finite-dimensional Artin algebra with unit

$$\mathbf{m}' := \mathbf{m} \oplus (\text{ground field}) \cdot \mathbf{1} \ .$$

In general, it is convenient to think about a commutative associative algebras without unit as about objects dual to spaces with base points. Algebra corresponding to a space with a base point is the algebra of functions vanishing at the base point.

2.1. Examples

There are many standard examples of differential graded Lie algebras and related moduli problems. We recall two cases:

1) Let X be a complex manifold. Define $\mathbf{g}^k$ for $k \geq 0$ as

$$\Gamma(X, \Omega^{0,k} \otimes T^{1,0})$$

with the differential equal to $\overline{\partial}$ and the Lie bracket coming from the cup-product on $\overline{\partial}$-forms and the usual Lie bracket on holomorphic vector fields. Then the deformation functor related with $\mathbf{g}$ is canonically equivalent to the usual deformation functor for complex structures on X.

2) Let A be an associative algebra. We define $\mathbf{g}^k$ for $k \geq -1$ as $Hom(A^{\otimes(k+1)}, A)$. The differential is the usual differential in the Hochschild complex, and the Lie bracket is the Gerstenhaber bracket. We would like to recall here the definition of these structures. Let F denote the free coassociative graded super coalgebra with counit cogenerated by the graded vector space $A[1]$, i.e. A endowed with the pure grading in degree -1. Then $\mathbf{g}$ is the Lie algebra of coderivations of F in the tensor category of $\mathbf{Z}$-graded super vector spaces. The associative product on A defines an element $m \in \mathbf{g}^1$ satisfying the equation $[m, m] = 0$. The differential d in $\mathbf{g}$ is defined as $ad(m)$. Again, the deformation functor related with g is equivalent to the usual deformation functor for algebraic structures.

3. L_∞-morphisms, L_∞-algebras and quasi-isomorphisms

Let $\mathbf{g}_1$ and $\mathbf{g}_2$ be two differential graded Lie algebras.

Definition. *A pre-L_∞-morphism f from $\mathbf{g}_1$ to $\mathbf{g}_2$ is an infinite sequence of linear maps $f = (f_1, f_2, \ldots)$ between $\mathbf{Z}$-graded vector spaces:*

$$f_1 : \mathbf{g}_1 \longrightarrow \mathbf{g}_2$$
$$f_2 : \wedge^2(\mathbf{g}_1) \longrightarrow \mathbf{g}_2[-1]$$
$$f_3 : \wedge^3(\mathbf{g}_1) \longrightarrow \mathbf{g}_2[-2]$$
$$\ldots$$

In formulas above the exterior power of $\mathbf{g}_1$ is taken in the tensor category of graded super vector spaces. The suffix $[n]$, $n \in \mathbf{Z}$ denotes the tensor product with the standard one-dimensional space endowed with the grading in degree $-n$. Thus, in plain terms we have a collection of linear maps between graded components

$$f_{(k_1,\ldots,k_n)} : \mathbf{g}_1^{k_1} \otimes \ldots \otimes \mathbf{g}_1^{k_n} \longrightarrow \mathbf{g}_2^{k_1+\ldots+k_n+(1-n)}$$

with the symmetry property

$$f_{(k_1,\ldots,k_n)}(\gamma_1\otimes\ldots\otimes\gamma_n) = -(-1)^{k_i k_{i+1}} f_{(k_1,\ldots,k_{i+1},k_i,\ldots,k_n)}(\gamma_1\otimes\ldots\otimes\gamma_{i+1}\otimes\gamma_i\otimes\ldots\otimes\gamma_n)\ .$$

As we see, in this definition we use only the structure of graded vector spaces on $\mathbf{g}_1, \mathbf{g}_2$.

With any graded vector space $\mathbf{g}$ we associate the cofree coassociative cocommutative graded superalgebra without counit cogenerated by $\mathbf{g}[1]$:

$$C(\mathbf{g}) := \bigoplus_{k=1}^{\infty} Sym^k(\mathbf{g}[1]) = \bigoplus_{k=1}^{\infty} \wedge^k(\mathbf{g})[k]\ .$$

Notice that in this definition we use $\mathbf{g}$ with the *reversed* parity.

Intuitively, we think of $C(\mathbf{g})$ as of an object corresponding to a formal supermanifold with a based point, possibly infinite-dimensional:

$$(\text{Formal neighborhood of zero in } \mathbf{g}[1],\ 0)\ .$$

The reason for this is that if $\mathbf{g}$ is finite-dimensional then the dual space to $C(\mathbf{g})$ is the algebra of formal power series vanishing at the origin on the super vector space $\mathbf{g}[1]$. Pre-L_∞-morphisms from $\mathbf{g}_1$ to $\mathbf{g}_2$ correspond to graded morphisms of coalgebras

$$C(\mathbf{g}_1)\longrightarrow C(\mathbf{g}_2)$$

and (intuitively) to maps between formal supermanifolds with based points.

Suppose now that $\mathbf{g}$ is a differential graded Lie algebra. Then it is easy to see that the differential and the Lie bracket in $\mathbf{g}$ define a differential Q, $Q^2 = 0$ of degree $+1$ of the coalgebra $C(\mathbf{g})$. In the case of ordinary Lie algebras (in degree 0) we get the standard reduced chain complex. In geometrical terms, we have an action of $(0|1)$-dimensional graded abelian Lie supergroup $\mathbf{A}^1[-1]$ on a formal supermanifold preserving the base point.

The chain complex of a differential graded Lie algebra gives a formal supermanifold with a flat structure in which the odd vector field Q generating $\mathbf{A}^1[-1]$-action has terms of degree 1 and 2 only in the Taylor expansion at the origin.

In paper [AKSZ] supermanifolds with the action of $\mathbf{A}^1[-1]$ are called Q-manifolds. By analogy, we introduce here the following

Definition. *A formal Q-manifold is a differential graded coalgebra $\mathcal{C}$ which is isomorphic as a graded coalgebra to $C(\mathbf{g})$ for some graded vector space $\mathbf{g}$.*

We want to stress that the specific isomorphism between $\mathcal{C}$ and $C(\mathbf{g})$ is *not* considered as a part of data. If we fix it, then we will obtain so called L_∞-algebra (or homotopy Lie algebras, see [HS2]).

Definition. *An L_∞-algebra is a graded vector space $\mathbf{g}$ and a differential Q of degree $+1$ on the graded coalgebra $C(\mathbf{g})$.*

The structure of an L_∞-algebra is given by the infinite sequence of Taylor coefficients Q_i of the odd vector field Q (coderivation of $C(\mathbf{g})$):

$$Q_1 : \mathbf{g}\longrightarrow\mathbf{g}[-1]$$

$$Q_2 : \wedge^2(\mathbf{g})\longrightarrow\mathbf{g}$$

$$Q_3 : \wedge^3(\mathbf{g})\longrightarrow\mathbf{g}[1]$$

$$\ldots$$

The condition $Q^2 = 0$ can be translated into an infinite sequence of quadratic constraints on polylinear maps Q_i. First of these constraints means that Q_1 is the differential of the graded space $\mathbf{g}$. The second constraint means that Q_2 is a skew-symmetric bilinear product on $\mathbf{g}$ compatible with Q_1 by the Leibnitz rule. The third constraint means that Q_2 satisfies the Jacobi identity up to homotopy given by Q_3, etc. . As we have seen, a differential graded Lie algebra is the same as an L_∞-algebra with $Q_3 = Q_4 = \ldots = 0$.

Nevertheless, I recommend to return to the geometric point of view and think in terms of formal Q-manifolds. This naturally leads to the following

Definition. *An L_∞-morphism between two L_∞-algebras is a morphism between corresponding differential graded cocommutative coalgebras.*

For the case of differential graded Lie algebras an L_∞-morphism can be identified with a pre-L_∞-morphism satisfying the following equation for any $n = 1, 2 \ldots$:

$$df_n(\gamma_1 \wedge \gamma_2 \wedge \ldots \wedge \gamma_n) - \sum_{i=1}^{n} \pm f_n(\gamma_1 \wedge \ldots \wedge d\gamma_i \wedge \ldots \wedge \gamma_n) =$$

$$= \frac{1}{2} \sum_{k,l \geq 1,\ k+l=n} \frac{1}{k!l!} \sum_{\sigma \in \Sigma_n} \pm [f_k(\gamma_{\sigma_1} \wedge \ldots \wedge \gamma_{\sigma_k}), f_l(\gamma_{\sigma_{k+1}} \wedge \ldots \wedge \gamma_{\sigma_n})] + \sum_{i<j} \pm f_{n-1}([\gamma_i, \gamma_j] \wedge \gamma_1 \wedge \ldots \wedge \gamma_n) \ .$$

Here are first two equations in the explicit form:

$$df_1(\gamma_1) = f_1(d\gamma_1)\,,$$

$$df_2(\gamma_1 \wedge \gamma_2) - f_2(d\gamma_1 \wedge \gamma_2) - (-1)^{deg(\gamma_1)} f_2(\gamma_1 \wedge d\gamma_2) = f_1([\gamma_1, \gamma_2]) - [f_1(\gamma_1), f_1(\gamma_2)] \ .$$

We see that f_1 is a morphism of complexes.

L_∞-morphisms generalize usual morphisms of differential graded Lie algebras. We call a L_∞-morphism f a quasi-isomorphism iff the first component f_1 induces isomorphism between cohomology groups of complexes $\mathbf{g}_1$ and $\mathbf{g}_2$.

The essence of the homotopy/deformation theory is contained in the following

Theorem. *Let f be an L_∞-morphism form $\mathbf{g}_1$ to $\mathbf{g}_2$. Assume that f is a quasi-isomorphism. Then there exists an L_∞-morphism form $\mathbf{g}_1$ to $\mathbf{g}_2$ inducing the inverse isomorphism between cohomology of complexes g_i, $i = 1, 2$. Also, for the case of differential graded algebras, L_∞-morphism f induces an equivalence between deformation functors associated with $\mathbf{g}_i$.*

The first part of this theorem shows that if $\mathbf{g}_1$ is quasi-isomorphic to $\mathbf{g}_2$ then $\mathbf{g}_2$ is quasi-isomorphic to $\mathbf{g}_1$, i.e. we get an equivalence relation.

The second part of the theorem works as follows. Any solution of the Maurer-Cartan equation in $\mathbf{g}_1$ depending formally on a parameter $\hbar$:

$$\gamma(\hbar) = \gamma_1 \hbar + \gamma_2 \hbar^2 + \ldots \in \mathbf{g}_1^1[[\hbar]],\ \ d\gamma(\hbar) + \frac{1}{2}[\gamma(\hbar), \gamma(\hbar)] = 0$$

produces a solution of the Maurer-Cartan equation in $\mathbf{g}_2$:

$$\tilde{\gamma}(\hbar) = \sum_{n=1}^{\infty} \frac{1}{n!} f_n(\gamma(\hbar) \wedge \ldots \wedge \gamma(\hbar)) \in \mathbf{g}_2^1[[\hbar]],\ \ d\tilde{\gamma}(\hbar) + \frac{1}{2}[\tilde{\gamma}(\hbar), \tilde{\gamma}(\hbar)] = 0 \ .$$

This theorem is essentially standard (see related results in [GM], [HS1]). My approach consists in the translation of all relevant notions to the geometric language of formal Q-manifolds. The main technical result in this approach is a (dual) version of Sullivan's theory of minimal models (see [S]).

In order to formulate it we introduce two notions:

Definition. *An L_∞-algebra $\mathbf{g}$ is called* **minimal** *if the first Taylor coefficient Q_1 of the differential Q vanishes.*

Definition. *An L_∞-algebra $\mathbf{g}$ is called* **linear contractible** *if all higher Taylor coefficients Q_k, $k \geq 2$ of the differential Q vanish, and the cohomology of the differential Q_1 on the garded space $\mathbf{g}$ vanish.*

Lemma (Minimal Model Theory). *Any L_∞-algebra $\mathbf{g}$ is L_∞-isomorphic to the product of a minimal L_∞-algebra $\mathbf{g}_{min}$ and a linear contractible L_∞-algebra $\mathbf{g}_{contr}$.*

This lemma can be proved by induction in degrees of terms of coefficients of Taylor expansions (see [K2]). From this lemma follows that the set of quasi-isomorphisms classes of L_∞-algebras is in one-to-one correspondence with the set of L_∞-equivalence classes of minimal algebras.

4. The main conjecture

Let X be a smooth manifold. We associate with it two differential graded Lie algebras. The first Lie superalgebra $D^*(X)$ is a subalgebra of the Hochschild complex of the algebra A of functions on X (see the last paragraph of Section 2). The space $D^n(X),\ n \geq -1$ consists of local Hochschild cochains $A^{\otimes(n+1)} \longrightarrow A$ given by polydifferential operators. In local coordinates (x_i) any element of D^n can be written as

$$\phi_0 \otimes \ldots \otimes \phi_n \mapsto \sum_{(I_0,\ldots,I_n)} C_{I_0,\ldots,I_n}(x) \cdot \partial^{I_0}(\phi_0) \ldots \partial^{I_n}(\phi_n)$$

where the sum is finite, I_k denote multi-indices, ∂^{I_k} denote corresponding partial derivatives, and ϕ_k and $C_{I_0,\ldots,I_n}$ are functions in (x_i).

The second differential graded Lie algebra $T^*(X)$ is the Lie superalgebra of polyvector fields on X:

$$T^n(X) = \Gamma(X, \wedge^{n+1} T_X),\ \ n \geq -1$$

endowed with the standard Schouten-Nijenhuis bracket and with the differential $d := 0$.

We have an evident map $f_1 : T^*(X) \longrightarrow D^*(X)$:

$$f_1 : (\xi_0 \wedge \ldots \wedge \xi_n) \mapsto \left(\phi_0 \otimes \ldots \otimes \phi_n \mapsto \frac{1}{n!} \sum_{\sigma \in \Sigma_n} sgn(\sigma) \prod_{i=1}^{n} \xi_{\sigma_i}(\phi_i) \right) .$$

Here ξ_i are vector fields on X and ϕ_i are functions. By a version of Kostant-Hochschild-Rosenberg theorem this map is a quasi-isomorphism of complexes.

Formality Conjecture. *There exists an L_∞-morphism f from $T^*(X)$ to $D^*(X)$ with the first term f_1 fixed as above.*

In other words, this conjecture claims that $T^*(X)$ and $D^*(X)$ are quasi-isomorphic differential graded Lie algebras. In analogous situation in rational homotopy theory (see [S]), a differential graded commutative algebra is called formal if it is quasi-isomorphic to its cohomology algebra endowed with zero differential. This explains the name of my conjecture.

Solutions of the Maurer-Cartan equation in $T^*(X)$ are exactly Poisson structures on X. The gauge group action is the action of the diffeomorphism group by conjugation. Thus, the conjecture implies that every Poisson manifold can be canonically quantized.

4.1. Reformulation

Here I present an attempt to reformulate the formality conjecture in some vague terms. This part is not essential for the rest of the paper and can be regarded as a deviation.

The Lie superalgebra $T^*(X)$ is the Lie superalgebra of functions on supermanifold $\Pi T^* X$ (the odd cotangent bundle to X) with respect to the odd Poisson bracket. Its quotient modulo the center (constant functions) coincides with the Lie superalgebra of hamiltonian vector fields on $\Pi T^* X$. In order to get a more neat formulation we can add one extra odd coordinate to $\Pi T^* X$ and get a supermanifold Y with an odd contact 1-form α. Lie superalgebra $T^*(X)$ coincides with the Lie superalgebra of infinitesimal automorphisms of (Y, α).

I propose to consider the differential graded Lie algebra $D^*(X)$ as the Lie algebra of infinitesimal automorphisms in homotopy sense of the triangulated category $D^b(A - mod)$ (or, better, A_∞-category, see [K]), where A is the algebra of functions on X. First of all, the truncated algebra $D^{\geq 0}(X)$ is responsible for automorphisms /deformations of A in the homotopy sense. When we add to it the term $D^{-1}(X) \simeq A$ then it means that we consider inner automorphims of A as trivial. Morally, it is ananlogous to the passage from groups to groupoids, i.e. to categories. I am sorry, but I can't be more precise there.

Thus, the formality conjecture means that there exists a natural construction of an A_∞-category from a supermanifold with an odd contact 1-form (and vice versa).

For me it looks as an odd version of the usual star-quantization of even symplectic manifolds. Constructions by De Wilde - Lecomte and Fedosov (see [DL], F]) of the star-quantization of symplectic manifolds show that with every symplectic manifold one can associate in a canonical way an equivalence class of an associative algebra $\mathcal{A}$ over $\mathbf{R}[[\hbar]]$. Unfortunately, the algebra $\mathcal{A}$ itself can not be constructed canonically. Nevertheless, after an additional work one can construct canonically an abelian category which is equivalent to the category of $\mathcal{A}$-modules (J. Bernstein, M.K., unpublished).

In the usual even situation the quantization of the *algebraic* symplectic manifold $\mathbf{R}^{2n}$ with the standard symplectic structure gives the algebra of polynomial differential operators on $\mathbf{R}^n$. The symplectic group acts by automorphisms of this algebra. In particular, the Fourier transform identifies algebras of polynomial differential operators on dual vector spaces. It seems that in the odd symplectic geometry it corresponds to the result of Beilinson and Bernstein-Gelfand-Gelfand on the equivalence (after small modifications) of bounded derived categories of modules over symmetric and exterior algebras (see [B] and [BBG]). Morally, triangulated categories for algebras of functions on $\mathbf{R}^{n|0}$ and on $\mathbf{R}^{0|n}$ are equivalent.

In paper [AKSZ] we proposed a Lagrangian for a topological two-dimensional sigma-model with the target space being an odd symplectic manifold. There are some heuristic arguments (see [K1]) showing that boundary conditions for topological field theories in two dimensions coupled with gravity form an A_∞-category. Thus, one can hope that the string theory can provide a proof of the formality conjecture.

5. Obstructions to formality

In this section we will show that if the formality conjecture is wrong then there exists a non-zero cohomology class in certain universal complex, the odd Graph Complex (various versions of the Graph Complex were introduced in [K2]). Direct calculations with this complex show that the appropriate cohomology groups vanish up to degree 6 in perturbation theory. It seems quite possible that all these cohomology groups of the odd Graph Complex vanish.

5.1. General obstruction theory

Let $\mathbf{g}$ be a differential graded Lie algebra and $\mathbf{g}_1$ be its algebra of cohomology considered as a differential graded Lie algebra (with zero differential). We are looking for a quasi-isomorphism between $\mathbf{g}$ and $\mathbf{g}_1$, i.e. asking whether $\mathbf{g}$ is formal, or not. Let us start to construct components f_i of an L_∞-morphism from $\mathbf{g}_1$ to $\mathbf{g}$ by induction. The first component can be constructed as a quasi-isomorphism of complexes $f_1 : \mathbf{g}_1 \longrightarrow \mathbf{g}$ which identifies graded Lie algebras $\mathbf{g}_1$ and $H^*(\mathbf{g})$.

We assume that we already constructed components $f_1, \dots, f_N$ satisfying first N equations on an L_∞-morphism:

$$\forall n \le N \quad df_n(\gamma_1 \wedge \ldots \wedge \gamma_n) = \sum_{i<j} \pm f_{n-1}([\gamma_i, \gamma_j] \wedge \gamma_1 \wedge \ldots \wedge \gamma_n) +$$

$$+\frac{1}{2} \sum_{k,l\ge 1,\ k+l=n} \frac{1}{k!l!} \sum_{\sigma\in\Sigma_n} \pm [f_k(\gamma_{\sigma_1} \wedge \ldots \wedge \gamma_{\sigma_k}), f_l(\gamma_{\sigma_{k+1}} \wedge \ldots \wedge \gamma_{\sigma_n})] =$$

$$= \left(\sum_{i<j} \pm f_{n-1}([\gamma_i, \gamma_j] \wedge \gamma_1 \wedge \ldots \wedge \gamma_n) + \sum_i \pm [f_1(\gamma_i), f_{n-1}(\gamma_1 \wedge \ldots \wedge \gamma_{i-1} \wedge \gamma_{i+1} \wedge \ldots \wedge \gamma_n)] \right) +$$

$$+\frac{1}{2} \sum_{k,l\ge 2,\ k+l=n} \frac{1}{k!l!} \sum_{\sigma\in\Sigma_n} \pm [f_k(\gamma_{\sigma_1} \wedge \ldots \wedge \gamma_{\sigma_k}), f_l(\gamma_{\sigma_{k+1}} \wedge \ldots \wedge \gamma_{\sigma_n})] \ .$$

Here we use the fact that the differential in $\mathbf{g}_1$ is equal to 0.

We want to solve analogous equation for $n = N+1$. It is easy to see that the right hand side of this equation is an element of $Ker(d) \subset \mathbf{g}$. This follows directly from all previous equations. If for all γ_i the r.h.s. belongs to $Im(d) \subset Ker(d)$, we are ready to construct f_{N+1}. The obstruction is a polylinear map:

$$\Phi : \wedge^{N+1}(\mathbf{g}_1) \longrightarrow \mathbf{g}_1[1-N] \ .$$

I claim that this map is cocycle of the graded Lie superalgebra $\mathbf{g}_1$ with the coefficients in the (twisted by $[1-N]$) adjoint representation. It means that

$$\sum_{i<j} \pm\Phi([\gamma_i,\gamma_j]\wedge\gamma_0\wedge\ldots\wedge\gamma_{N+1}) + \sum_i \pm[\gamma_i, \Phi(\gamma_0\wedge\ldots\wedge\gamma_{i-1}\wedge\gamma_{i+1}\wedge\ldots\wedge\gamma_{N+1})] = 0\ .$$

Again, it follows directly from all previous equations.

Let us assume that the cohomology class of this cocycle is zero:

$$\Phi(\gamma_0\wedge\ldots\wedge\gamma_N) = \sum_{i<j} \pm\Psi([\gamma_i,\gamma_j]\wedge\gamma_0\wedge\ldots\wedge\gamma_N) + \sum_i \pm[\gamma_i, \Psi(\gamma_0\wedge\ldots\wedge\gamma_{i-1}\wedge\gamma_{i+1}\wedge\ldots\wedge\gamma_N)]$$

for some linear map $\Psi : \wedge^N(\mathbf{g}_1)\longrightarrow\mathbf{g}_1[1-N]$.

Then, if we replace f_N by

$$\tilde{f}_N : \gamma_1\wedge\ldots\wedge\gamma_N \mapsto f_N(\gamma_1\wedge\ldots\wedge\gamma_N) + f_1(\Psi(f_N))$$

we obtain a new collection of maps $(f_1,\ldots,\tilde{f}_N)$ with vanishing obstruction to the existence of f_{N+1}.

We see that the obstruction to the formality lies in cohomology groups which can be loosely denoted by $H^N(\mathbf{g}_1,\mathbf{g}_1[1-N])$. These cohomology groups are cohomology of complexes consisting in skew-symmetric polylinear maps (in the tensor category of graded vector spaces) from $\mathbf{g}_1$ to $\mathbf{g}_1[1-N]$ with the differential imitating the usual Chevalley-Eilenberg differential.

5.2. Polyvector fields and the Graph Complex

Let us apply the general result to the Lie superalgebra of polyvector fields in the flat space $\mathbf{R}^d$. As a graded vector space this superalgebra $T^*(\mathbf{R}^d)$ can be identified with the space of functions on supermanifold $\mathbf{R}^{d|d}$ with coordinates $x_i,\xi^i,\ i=1,\ldots,d$. The first group of coordinates, (x_i) is pure even, in degree 0, the second group (ξ^i) is purely odd, in degree +1. Loosely speaking, ξ^i corresponds to $\partial/\partial x_i$.

One can show that the first non-vanishing obstruction class, if it exists, should be in certain sense stable (independent of the dimension). There is a natural class of stable cochains of $T^*(\mathbf{R}^d)$ associated with finite graphs. I claim that the obstruction class to the formality conjecture comes from a cohomology class in the (odd) Graph Complex. The proof of this statement is too long and technical to include there. Thus, the reader should consider the rest of this section only as an announcement.

Here follows the description of cochains associated with graphs. Let Γ be a finite non-oriented graph (= finite 1-dimensional CW-complex) without multiple edges. If we enumerate vertices of this graph by $\{1,\ldots,n\}$ then this graph is defined by its incidence matrix $M=(M_{kl})$ which is symmetric with entries in $\{0,1\}$.

Let us also enumerate the set of edges of Γ by $\{1,\ldots,m\}$. The corresponding cochain of $T^*(\mathbf{R}^d)$ is the obtained by antisymmetrization from the following polylinear map $\mathbf{g}_1^{\otimes n}\longrightarrow\mathbf{g}_1[n-m-1]$:

$$\phi_1\otimes\ldots\otimes\phi_n \mapsto \left(\left(\prod_{k\le l,\ M_{kl}=1}\Omega_{kl}\right)(\phi_1\otimes\ldots\otimes\phi_n)\right)_{diag}\ .$$

Here on the r.h.s. the tensor product of functions ϕ_i on $\mathbf{R}^{d|d}$ is considered as a function on $\left(\mathbf{R}^{d|d}\right)^n$. The ste of factors in the produact of Ω_{kl} is in one-to-one corespondence with the ste of edges of Γ. Differential operator Ω acts on functions on $\mathbf{R}^{d|d}\times\mathbf{R}^{d|d}$ by the formula

$$\Omega = \sum_i \left(\frac{\partial}{\partial x_i}\otimes\frac{\partial}{\partial\xi^i} + \frac{\partial}{\partial\xi^i}\otimes\frac{\partial}{\partial x_i}\right)\ .$$

Operator Ω_{kl} is a copy of Ω acting on the product of the k-th and the l-th factor of $\left(\mathbf{R}^{d|d}\right)^n$ for $k\neq l$. For $k=l$ we define Ω_{kk} as the differential operator on $\mathbf{R}^{d|d}$

$$\sum_i \frac{\partial^2}{\partial x_i\partial\xi_i}$$

acting along the k-th factor of $\left(\mathbf{R}^{d|d}\right)^n$.

Operators Ω_{kl} are odd anti-commuting differential operators on $\left(\mathbf{R}^{d|d}\right)^n$. In order to specify the meaning of their composition we use the enumeration of the set of edges of Γ (up to an even permutation). Then we restrict the result of the application of the composite operator to the tensor product of functions ϕ_k to the diagonal, and obtain again a function on $\mathbf{R}^{d|d}$.

Polylinear maps $\mathbf{g}_1^{\otimes n} \longrightarrow \mathbf{g}_1$ obtained in this way are not antisymmetric in general. After antisymmetrization some graphs give automatically zero cochain. The result of the antisymmetrization does not depend on the enumeration of vertices. Also, a boundary of the cochain given by a graph is obtained from a universal linear combination of graphs.

Now we are going to describe the part of the graph complex relevant for the formality conjecture.

Definition. *A finite graph is called* **good** *if it is nonempty, connected, nonseparable, has no multiple edges, and the valency (degree) of each vertex is greater than or equal to* 3. *The condition of nonseparability means that the complement to any vertex is connected.*

It follows from the definition that any good graph has no simple loops. In terms of the incidence matrix it means that all diagonal entries are 0. The simplest good graph has 4 vertices, it is the complete graph (the skeleton of a tetrahedron). For a graph Γ we define a one-dimensional vector space $Or(\Gamma)$ over $\mathbf{Q}$. It is the top exterior power of the vector space $\mathbf{Q}^{edges(\Gamma)}$ spanned by the set of edges of Γ. The automorphism group of Γ acts on the space $Or(\Gamma)$. If this action is non-trivial, i.e. there exists an automorphism acting by (-1) on $Or(\Gamma)$, then the corresponding cochain of the Lie superalgebra of polyvector fields vanishes.

The Graph complex will be bi-graded, with the differential of degree $(+1, +1)$.

The graded component $G^{n,m}$ is defined as

$$\bigoplus_{classes\ \ of\ \ \Gamma} (Or(\Gamma))_{Aut(\Gamma)} \ .$$

Here the sum is taken over all isomorphism classes of good graphs with n vertices and m edges. For each isomorphism class we choose a representative and take the space of coinvariants. This space does not depend on the choice of a representative.

It follows form the definiton of the Graph complex that every good graph Γ with enumerated edges defines an element in $G^{n,m}$. We denote this element by $[\Gamma, enum]$ where $enum$ denotes the choosen enumeration of edges. We are going to write a formula for $d[\Gamma, enum]$.

Let v be a vertex of graph Γ of degree ≥ 4. Denote by S_v the set of edges connected to v. Let us represent S_v as a disjoint union of two sets of cardinality ≥ 2: $S_v = S_v^1 \sqcup S_v^2$. With such a structure we associate a new good graph $\Gamma' = \Gamma'(v, S_v^1, S_v^2)$. This graph has $n+1$ vertices and $m+1$ edges.

Fisrt of all, we remove vertex v and all edges from S_v from Γ. Then we add two new vertices v_1 adn v_2. Edges from the set S_v^1 we connect to v_1 (and to the same vertex in $\Gamma \setminus \{v\}$ as it was in Γ), edges from S_v^2 connect to v_2, both new vertices v_1 and v_2 connect by a new edge e. We obtain graph Γ'. The set of edges of Γ' is obtained from the set of edges of Γ by adding a new element e. Thus, we get an enumeration $enum'$ of edges of Γ' by attaching number $m+1$ to the edge e.

The formula for the differential in the Graph Complex is

$$d[\Gamma, enum] = \sum_{(v, S_v^1, S_v^2)} [\Gamma'(v, S_v^1, S_v^2), enum'(v, S_v^1, S_v^2)] \ .$$

The square of this operator is equal to zero.

Claim. *Let us assume that the formality conjecture for flat spaces* $X = \mathbf{R}^d$, $d = 1, 2, \ldots$ *holds up to the* $(n-1)$*-st term in the perturbation theory. Then the possible obstruction on to the formality on the* n*-th step comes from a cohomology class of the graph complex in degree* $(n, 2n-3)$.

Elementary counting shows that $G^{n,2n-3} = 0$ for $n \leq 6$. Thus, we get a confirmation of the formality conjecture with a great precision.

In fact, any good 3-valent graph gives a cohomology class of the Graph Complex. Thus one can expect plenty of non-trivial cohomology classes in degrees (k, l) where $l/k = 3/2$. It seems quite plausible that degrees in which cohomology is non-trivial are close to the line $l/k = 3/2$ and there is no cohomology in degrees $(n, 2n-3)$ (close to the line $l/k = 2$) at all!

5.3. A flow on the space of Poisson manifolds

If we believe in the formality conjecture, we can ask ourselves whether the desired quasi-isomorphism is unique up to homotopy, or not. The corresponding cohomology groups of the graph complex have degrees $(n, 2n-2)$. I am aware of only one such a class, it corresponds to simplest good graph, the complete graph with 4 vertices (and 6 edges). This class gives a remarkable vector field on the space of bi-vector fields on $\mathbf{R}^d$. The evolution with respect to the time t is described by the following non-linear partial differential equation:

$$\frac{d\alpha}{dt} := \sum_{i,j,k,l,m,k',l',m'} \frac{\partial^3 \alpha_{ij}}{\partial x_k \partial x_l \partial x_m} \frac{\partial \alpha_{kk'}}{\partial x'_l} \frac{\partial \alpha_{ll'}}{\partial x'_m} \frac{\partial \alpha_{mm'}}{\partial x'_k} \left(\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \right)$$

where $\alpha = \sum_{i,j} \alpha_{ij}(x) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$ is a bi-vector field on $\mathbf{R}^d$.

A priori we can guarantee the existence of a solution of the evolution only for small times and real-analytic initial data.

More generally, this class produces a formal one-parameter group of L_∞-automorphisms of $T^*(\mathbf{R}^d)$.

It follows from general properties of cohomology that 1) this evolution preserves the class of (real-analytic) Poisson structures, 2) if two Poisson structures are conjugate by a real-analytic diffeomorphism then the same will hold after the evolution. Thus, our evolution operator is essentially intrinsic and does not depend on the choice of coordinates.

In fact, I cheated a little bit. In the formula for the vector field on the space of bivector fields which one get from the tetrahedron graph, an additional term is present. This term is equal (up to a numerical factor) to

$$\sum_{i,j,k,l,m,n,p,r} \frac{\partial^2 \alpha_{ij}}{\partial x_k \partial x_l} \frac{\partial^2 \alpha_{km}}{\partial x_n \partial x_p} \frac{\partial \alpha_{nl}}{\partial x_r} \frac{\partial \alpha_{rp}}{\partial x_j} \left(\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_m} \right) .$$

It is possible to prove formally that if α is a Poisson bracket, i.e. if $[\alpha, \alpha] = 0 \in T^2(\mathbf{R}^d)$, then the additional term shown above vanishes.

I tried in vain to find any example where my evolution operator really changes the conjugacy class of a Poisson bracket. The problem is that principal examples of Poisson brackets have coefficients which are either linear (Kirillov bracket), or quadratic polynomials (classical Sklyanin algebras). In the evolution I use third derivatives. Also, in dimension $d = 2$ the direct calculation shows that the evolution operator gives a conjugation of bivector field α by a vector field whose coefficients are differential polynomials in coefficients of α. Nevertheless, it is easy to check that in higher dimensions it is impossible to obtain a formal proof of triviality of the evolution operator. I don't know whether one can construct the evolution operator on arbitrary Poisson manifolds, not necessarily embeddable into the Euclidean space of the same dimension.

The formality conjecture implies that there should be analogous (formal) evolution operator on equivalence classes of non-commutative algebras close to algebras of functions on smooth manifolds.

As a side remark, I would like to mention that the cohomology of the odd Graph Complex form a Lie superalgebra which acts on the space of equivalence classes of germs of QP-manifolds (see [AKSZ] for the definition).

6. Comments and further evidences

In the formulation of the formality conjecture I didn't express all my wishes. First of all, graded components $f_{(k_1,\ldots,k_n)}$ of the desired L_∞-morphism f form $T^*(X)$ to $D^*(X)$ should be given by local expressions, i.e. by polydifferential operators in coefficients of polyvector fields.

Also, it would be nice to formulate the conjecture using the language of sheaves:

6.1. Sheaves

Differential graded Lie algebras $D^*(X)$ and $T^*(X)$ are global sections of natural sheaves D^* and T^*. By "naturalness" I mean that sheaves are defined in a coordinate-independent way. Sheaves D^* and T^* can be defined not only for C^∞-manifolds, but also for complex-analytic manifolds and for algebraic manifolds defined over a field of characteristic zero. One can just hope that there is a natural sheaf $\mathcal{X}^*$ of differential graded Lie algebras on manifolds (in a wide sense) with natural morphisms which are quasi-isomorphisms, to both sheaves D^* and T^*.

6.2. Cup-products on the tangent cohomology

If $\mathbf{g}$ is a differential graded Lie algebra and $\gamma \in \mathbf{g} \otimes \mathbf{m}$ is a solution of the Maurer-Cartan equation, where $\mathbf{m}$ is an Artin algebra, then one can define the tangent cohomology $T^*(\gamma)$ responsible for the deformation of γ. This is the cohomology group of the complex $\mathbf{g} \otimes \mathbf{m}$ endowed with the differential $d' := d + ad(\gamma)$. Cohomology groups are modules over $\mathbf{m}$. A more delicate invariant is the isomorphism class of this complex in the derived category of $\mathbf{m}$-modules. Again, any quasi-isomorphism induces the isomorphism of the tangent cohomology.

Let (X, α) be a Poisson manifold, i.e. $\alpha \in T^1(X)$, $[\alpha, \alpha] = 0$. Thus we get a solution of the Maurer-Cartan equation in $T^*(X)$ depending on formal variable $\hbar$:

$$\alpha(\hbar) = \alpha \cdot \hbar \in T^1[[\hbar]] \ .$$

Assuming the formality conjecture we get a solution of the Maurer-Cartan equation in $D^*(X)$, i.e. an associative algebra $\mathcal{A}$ over $\mathbf{R}[[\hbar]]$ which is isomorphic to $C^\infty(X)[[\hbar]]$ as $\mathbf{R}[[\hbar]]$-module.

On both tangent cohomology for $T^*(X)$ and for $D^*(X)$ there is a natural cup-product of degree $+1$.

For the the Lie superalgebra $T^*(X)$ we use just the usual cup-product of polyvector fields. For the case of $D^*(X)$ we use the interpretation of tangent cohomology (Hochschild cohomology) as the Ext-groups shifted by [1]:

$$Ext^*_{\mathcal{A} \otimes_{\mathbf{R}[[\hbar]]} \mathcal{A}^o - mod}(\mathcal{A}, \mathcal{A})[1] \ .$$

Then the Yoneda product gives a cup-product in tangent cohomology.

The tangent cohomology for the graded Lie algebra $T^*(X)$ coincides (after tensoring $\otimes_{\mathbf{R}[[\hbar]]} \mathbf{R}((\hbar))$) with the tensor product of $\mathbf{R}((\hbar))$ with the Poisson cohomology of (X, α) shifted by [1]. The Poisson cohomology $HP^*(X, \alpha)$ is defined as the cohomology of the complex $T^*(X)[-1]$ endowed with the differential $ad(\alpha)$.

Conjecture about cup-products. *Assuming the formality conjecture, the cup product on tangent cohomology for a solution of the Maurer-Cartan equation in $T^*(X)$ coincides with the cup-product on the tangent cohomology for the corresponding solution in $D^*(X)$.*

I am aware about one non-trivial example, confirming this conjecture. Namely, let X be the vector space dual to a Lie algebra $\mathbf{g}$ and α will be the usual Kirillov-Poisson structure on $X = \mathbf{g}^*$. The space X is considered here as an algebraic manifold, not a C^∞-manifold. I claim that the quantization of this Poisson manifold should be the universal enveloping algebra $U_\hbar(\mathbf{g})$ where the variable $\hbar$ is used as the deformation parameter:

$$U_\hbar(\mathbf{g}) = \text{algebra over } \mathbf{R}[[\hbar]] \text{ generated by } \mathbf{g}/\{\text{relations } X \cdot Y - Y \cdot X = \hbar[X, Y]\} \ .$$

The reason for this claim is based on certain considerations in invariant theory. Namely, the deformation $U_\hbar(\mathbf{g})$ should correspond to a 1-parameter family of Poisson structures on $X = \mathbf{g}^*$. Using $GL(\mathbf{g})$-invariant expressions in coefficients of the structure tensor of the Lie algebra $\mathbf{g}$ one can get only the standard linear Kirillov bracket. There is no invariant way to construct a polylinear map $\wedge^2(\mathbf{g}) \longrightarrow Sym^k(\mathbf{g})$ for $k \geq 2$, i.e. no way to get higher coefficients in the Taylor expansion for a Poisson bracket.

The Poisson cohomology in degree 0 of (X, α) is equal to the algebra of functions on X invariant under the coadjoint action. The Hochshild cohomology of $U(\mathbf{g})$ in degree 0 is the center $\mathcal{Z}(U(\mathbf{g}))$ of the universal enveloping algebra. Of course, in the definition of tangent cohomology we should include variable $\hbar$ into the game.

It is well-known that vector spaces $HP^0(X,\alpha)$ and $Z(U(\mathbf{g}))$ can be canonically identified:

$$HP^0(X,\alpha) \simeq \bigoplus_{k=0}^{\infty} \left(Sym^k(\mathbf{g})\right)^{\mathbf{g}} \simeq (U(\mathbf{g}))^{\mathbf{g}} \simeq \mathcal{Z}(U(\mathbf{g})) \ .$$

Nevertheless, the most naive identification does not preserve the algebra structure.

Fortunately, a long time ago M. Duflo (see [D]) based on Kirillov 's "universal formula" for characters of representations of Lie groups, constructed a different map $HP^0(X,\alpha) \longrightarrow \mathcal{Z}(U(\mathbf{g}))$ which gives an isomorphism of algebras. For the case of semi-simple Lie algebras it is the Harish-Chandra isomorphism. Strictly speaking, the construction of Duflo was done only for solvable Lie algebras. Nevrtheless, if one include the formal parameter $\hbar$, the formula became universal and didn't require the classification theory of Lie algebras. It seems that it works for finite-dimensional Lie algebras in arbitrary tensor categories in characteristic zero, although I am not aware of a general proof. Coefficients in Duflo-Kirillov formulas are quite non-trivial, they are equal essentially to the Bernoulli numbers.

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